Calculus on Power Series

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1 Differentiation and Integration

One motivation for studying power series is that some functions can be expressed as a power series, and power series enjoy some good properties regarding differentiation and integration. So by expressing a given function in terms of power series, we can obtain some calculus properties of the function.

We will start with the formula

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

for $|x| < 1$.

Example 1. *If we substitute x by $-x$ in the above formula, we get*

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$$

for $|-x| < 1$, i.e., $|x| < 1$.

A further substitution of x by x^2 leads to

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

for $|x| < 1$.

Example 2. *For the function $f(x) = \frac{1}{x-2}$, we write it as*

$$f(x) = \frac{1}{x-2} = -\frac{1}{2} \times \frac{1}{1-\frac{x}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty}(\frac{x}{2})^n = \sum_{n=0}^{\infty} -\frac{1}{2^{n+1}} x^n$$

for $|\frac{x}{2}| < 1$, i.e., $|x| < 2$.

Example 3. *Find a power series representation for the function* $f(x) = \frac{x^2}{5+x^3}$.

$$f(x) = \frac{x^2}{5+x^3} = \frac{x^2}{5} \times \frac{1}{1+\frac{x^3}{5}} = \frac{x^2}{5} \times \sum_{n=0}^{\infty}(-\frac{x^3}{5})^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}} x^{3n+2}$$

for $|-\frac{x^3}{5}| < 1$, *i.e.*, $|x| < \sqrt[3]{5}$.

An interesting property of power series regarding differentiation and integration is the following:

Theorem 4. *If* $f(x) = \sum_{n=0}^{\infty} c_n(x - x_0)^n$ *has radius of convergence* $R > 0$ *(possibly* $R = \infty$*), then* $f(x)$ *is differentiable and integrable on* $(x_0 - R, x_0 + R)$*, with:*

1. $f'(x) = \sum_{n=1}^{\infty} c_n n(x - x_0)^{n-1} = c_1 + 2c_2(x - x_0) + 3c_3(x - x_0)^2 + ...$

2. $\int f(x)\, dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x - x_0)^{n+1} + C = c_0(x - x_0) + \frac{c_1}{2}(x - x_0)^2 + \frac{c_2}{3}(x - x_0)^2 + ... + C$

That is, the differentiation and integration can be done termwisely.

And the radius of convergence of $f'(x)$, $\int f(x)\, dx$ *are also* R.

Example 5. *We know that* $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ *with radius of convergence* $R = 1$*, so differentiating the function, we get*

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$$

on $(-1, 1)$.

On the other hand, if we integrating the function, we get

$$-\ln(1 - x) + C = \int \frac{1}{1-x}\, dx = \int \sum_{n=0}^{\infty} x^n\, dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

on $(-1,1)$. *Let* $x=0$, *we get* $-\ln(1)+C=0$, *so* $C=0$, *we conclude*

$$\ln(1-x)=\sum_{n=0}^{\infty}-\frac{x^{n+1}}{n+1}=\sum_{n=1}^{\infty}-\frac{x^n}{n}$$

on $(-1,1)$.

A corollary from the above computation is that

$$\ln(1+x)=\sum_{n=1}^{\infty}-\frac{(-x)^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+...$$

on $(-1,1)$

Example 6. *We know that* $(\tan^{-1}x)'=\frac{1}{1+x^2}$, *so*

$$\tan^{-1}x+C=\int\frac{1}{1+x^2}\,dx=\int\sum_{n=0}^{\infty}(-1)^n x^{2n}\,dx=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$$

Let $x=0$, *we obtain* $C=0$, *so*

$$\tan^{-1}x=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+...$$

on $(-1,1)$.

Remark 7. Note that when $x=1$, the power series $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ is also convergent by Alternating Convergence Theorem. It can be shown with the knowledge of Mathematical Analysis that $\lim\limits_{x\to1^-}\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}=\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$, so:

$$\frac{\pi}{4}=\tan^{-1}1=\lim_{x\to1^-}\tan^{-1}x=\lim_{x\to1^-}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

i.e.,

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...$$

2 Taylor Series and Maclaurin Series

This section we will answer the following question: Given a function $f(x)$, is it possible to express it as a power series? We will start by looking for some necessaries conditions to formulate possible candidates, and then looking for some criteria to determine if the candidate power series indeed equals to the given function.

Theorem 8. *If $f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$ on $(a - R, a + R)$, where R is the radius of convergence of the power series, then*

$$c_n = \frac{f^{(n)}(x_0)}{n!}$$

Proof. We apply the differentiation rule for power series n-times, followed by evaluating the resulting formula at $x = x_0$ to obtain the theorem. □

This Theorem implies that if a given function $f(x)$ can be expressed as a power series in terms of $x - x_0$, then the only possible choice of the power series is

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

The power series $T(x)$ is called the **Taylor Series** of f at x_0. When $x_0 = 0$, the series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} x^n$ is called the **Maclaurin Series** of f at x_0.

After finding the candidate, that is, the Taylor series, the question to ask is natural: does $f(x)$ equal to $T(x)$? within the radius of convergence of $T(x)$?

Note that $f(x) = T(x) = \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ by definition is

$$f(x) = \lim_{N \to \infty} \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Let $T_N(x) = \displaystyle\sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ (called the n-th degree Taylor polynomial) and $R_N(x) = f(x) - T(x)$, the above discussion leads to the following Theorem:

Theorem 9. *If $f(x) = T_N(x) + R_N(x)$, and $\lim\limits_{N\to\infty} R_N(x) \to 0$ on the interval $(x_0 - R, x_0 + R)$, then $f(x) = T(x)$ on $(x_0 - R, x_0 + R)$.*

In practice, there is a formula to help us determine whether $\lim\limits_{N\to\infty} R_N(x) = 0$:

Theorem 10 (Taylor's Formula)**.** *If $f^{(N+1)}(x)$ exists on an interval I and $x_0 \in I$, then for any $x \neq x_0$ in I, there exists z strictly between x and x_0 such that*

$$R_N(x) = \frac{f(N+1)(z)}{(N+1)!}(x - x_0)^{N+1}$$

Example 11. *Let $f(x) = e^x$, choose $x_0 = 0$, then*

$$R_N(x) = \frac{f^{N+1}(z)}{(N+1)!}(x - x_0)^{N+1} = \frac{e^z}{(N+1)!}x^{N+1}$$

For any real number x,

$$\lim_{N\to\infty} R_N(x) = \lim_{N\to\infty} \frac{e^z}{(N+1)!}x^{N+1} = 0$$

since $\lim_{N\to\infty} \frac{x^{N+1}}{(N+1)!} = 0$ *and* $|e^z| \leq e^{|x|}$.

So we conclude $f(x) = e^x$ equals to its Taylor series for all x:

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{e^0}{n!}x^n = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

on $(-\infty, +\infty)$.

In particular, when $x = 1$, we get

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

We can make use of this Maclaurin Series to write $f(x) = e^x$ as the Taylor series at $x_0 = 2$:

$$e^{x-2} = \sum_{n=0}^{\infty} \frac{1}{n!}(x-2)^n$$

which becomes:

$$e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$$

Example 12. *We are going to write $f(x) = \sin x$ as a Maclaurin Series.*

Direct computation shows that

$$f^{(n)}(x) = \begin{cases} \sin x \ \textit{if } n = 4k \\ \cos x \ \textit{if } n = 4k+1 \\ -\sin x \ \textit{if } n = 4k+2 \\ -\cos x \ \textit{if } n = 4k+3 \end{cases}$$

So

$$f^{(n)}(0) = \begin{cases} 0 \ \textit{if } n = 4k \\ 1 \ \textit{if } n = 4k+1 \\ 0 \ \textit{if } n = 4k+2 \\ -1 \ \textit{if } n = 4k+3 \end{cases}$$

The reminder is

$$R_N(x) = \frac{f^{(N+1)}(z)}{(N+1)!} x^{N+1}$$

Observe that $|R_N(x)| = |\frac{f^{(N+1)(z)}}{(N+1)!} x^{N+1}| \le \frac{|x|^N}{(N+1)!}$, *and for each* x, $\lim_{N\to\infty} \frac{|x|^N}{(N+1)!} = 0$, *we get*

$$\lim_{N\to\infty} R_N = 0$$

So $f(x)$ *equals to its Maclaurin series for all* x*:*

$$\sin x = \sum_{N=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

Taking the derivative of the above expression, we also get

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

Next we will find the Maclaurin series of functions of the form $f(x) = (1+x)^k$, where k is a constant real number.

By computation, $f'(x) = k(1+x)^{k-1}$, $f''(x) = k(k-1)(1+x)^{k-2}$,..., $f^{(n)}(x) = k(k-1)(k-2)...(k-n+1)(1+x)^{k-n}$.

Evaluating at $x = 0$, we get: $f(0) = 1$, $f'(0) = k$, $f''(0) = k(k-1)$,..., $f^{(n)}(0) = k(k-1)...(k-n+1)$.

If we write $\binom{k}{n} = \begin{cases} 1 \text{ if } n = 0 \\ \frac{k(k-1)...(k-n+1)}{n!} \text{ if } n \geq 1 \end{cases}$

Then the Maclaurin series corresponding to $f(x) = (1+x)^k$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

called **binomial series**. And the coefficients $\binom{k}{n}$ are called **binomial coefficients**.

If k is a nonnegative integer, then the binomial series is finite, so it converges for all $x \in \mathbb{R}$.

If k is not a nonnegative integer, the series has infinitely many nonzero terms, and the radius of convergence of the binomial series can be computed as follows:

$$R = \lim_{n\to\infty} |\frac{a_n}{a_{n+1}}| = \lim_{n\to\infty} |\frac{\binom{k}{n}}{\binom{k}{n+1}}| = \lim_{n\to\infty} |\frac{n+1}{k-n}| = 1$$

Theorem 13.

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

for $|x| < 1$.

Example 14. *Find the Maclaurin series for* $f(x) = \frac{1}{\sqrt[3]{8-x}}$

$$\begin{aligned} f(x) &= \frac{1}{\sqrt[3]{8-x}} = \frac{1}{2\sqrt[3]{1-\frac{x}{8}}} = \frac{1}{2}(1-\frac{x}{8})^{-\frac{1}{3}} = \frac{1}{2}\sum_{n=0}^{\infty} \binom{-\frac{1}{3}}{n} (-\frac{x}{8})^n \\ &= \frac{1}{2}(1 + \sum_{n=1}^{\infty} \frac{(-\frac{1}{3})(-\frac{1}{3}-1)...(-\frac{1}{3}-n+1)}{n!}(-\frac{1}{8})^n x^n) \\ &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 \times 4 \times 7... \times (3n-2)}{n!3^n 2^{3n+1}} x^n \end{aligned}$$

And the series converges when $|-\frac{x}{8}| < 1$, i.e., $|x| < 8$, the radius of convergence is $R = 8$.

Example 15. *Evaluate* $\lim_{x\to 0} \frac{\sin x - x}{x^3}$.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin x - x}{x^3} &= \lim_{x\to 0} \frac{-\frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}...}{x^3} \\
&= \lim_{x\to 0} -\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!}... \\
&= -\frac{1}{6}
\end{aligned}$$

We can take the product and quotient of power series in a similar way to those of polynomials:

Example 16. *Find the first three terms in the Maclaurin series for* $e^x \ln(1+x)$ *on* $(0, 2)$.

$$\begin{aligned}
e^x \ln(1+x) &= (1 + x + \frac{x^2}{2} + \frac{x^3}{6} + ...)(x - \frac{x^2}{2} + \frac{x^3}{3} - ...) \\
&= 1 \times (x - \frac{x^2}{2} + \frac{x^3}{3}) + x(x - \frac{x^2}{2}) + \frac{x^2}{2}(x) + \textit{higher order terms} \\
&= x + \frac{x^2}{2} + \frac{x^3}{3} + \textit{higher order terms}
\end{aligned}$$

Example 17. *Find the first three terms of the Maclaurin expansion of* $f(x) = \sec x$ *on* $(-\frac{\pi}{2}, \frac{\pi}{2})$.

$$\begin{aligned}
\sec x = \frac{1}{\cos x} &= \frac{1}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...} \\
&= \frac{1}{1 - (\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!}...)} \\
&= 1 + (\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!}...) + (\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!}...)^2 + (\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!}...)^3 + ... \\
&= 1 + \frac{x^2}{2} - \frac{x^4}{24} + (\frac{x^2}{2})^2 + \textit{higher order terms} \\
&= 1 + \frac{3}{4}x^2 - \frac{x^4}{24} + \textit{higher order terms}
\end{aligned}$$